Math 120A
Differential Geometry

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [4pts.] Define an orientable surface.

(b) [5pts.] Prove that the torus with the parametrization

$$\sigma(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$$

applied to appropriate domains is an orientable surface. [Hint: You can argue this without doing a computation.]

(c) [1pts.] Comment on what this means about the shape of the torus.

Problem 2.

(a) [5pts.] Define an allowable surface patch for a smooth surface S.

(b) [5pts.] Let $f(x, y)$ be any smooth function from an open set $U \subset \mathbb{R}^2$ to $\mathbb{R}$. Prove that the graph $S = \{(x, y, f(x, y))\}$ is a smooth surface covered by a single allowable surface patch, and find an the tangent plane of f at an arbitrary point $(x_0, y_0, f(x_0, y_0))$.

Problem 3.

(a) [5pts.] Define a quadric.

(b) [5pts.] Verify that the solution set S to $16x^2 + 24xy + 9y^2 + 25x = 1$ is a smooth surface. What type of surface is it?

Problem 4.

(a) [5pts.] Define a vertex of a regular simple closed curve $\gamma(t)$ in $\mathbb{R}^2$.

(b) [5pts.] Show that there is no simple closed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with turning angle $\phi(t)$ equal to a polynomial of degree > 1. [Hint: What is $\ddot{\phi}(t)$?]

Problem 5.

Let $f : S \to S$ be the smooth map from the unit cylinder to itself which is the restriction of the map

$$\begin{aligned} \mathbb{R}^3 &\to \mathbb{R}^3 \\ (x, y, z) &\mapsto \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}, z^2 \right) \end{aligned}$$

Equivalently, this is the map $(r\cos\theta, r\sin\theta, z) \mapsto (r\sin(2\theta), r\cos(2\theta), z^2)$.

(a) [5pts.] Choose an atlas for the unit cylinder and find the corresponding matrix representations of $D_{(1,0,3)}f$ and $D_{(0,1,0)}f$.

(b) [5pts.] Decide whether this map is a local diffeomorphism. After proving your answer, give a one-sentence geometric description of why or why not.